IRRATIONALITY EXPONENTS OF NUMBERS RELATED WITH CAHEN'S CONSTANT

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This is a report on recent results of myself jointly with Duverney [7] on irrationality exponents of numbers related with Cahen's constant.

For a real number α, the irrationality exponent $\mu(\alpha)$ is defined by the greatest lower bound of the set of numbers μ for which the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\mu}} \tag{0.1}$$

has only finitely many rational solutions p/q, or equivalently the least upper bound of the set of numbers μ for which the inequality (0.1) has infinitely many solutions. If α is irrational, then $\mu(\alpha) \geq 2$. If α is a real algebraic irrationality, then $\mu(\alpha) = 2$ by Roth's theorem. If $\mu(\alpha) = \infty$, then α is called a Liouville number.

The main theorem of this paper, [7] stated below gives lower and upper bounds for the irrationality exponents of continued fractions.

We employ the usual notations for continued fractions :

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{+ b_n} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}} = \frac{A_n}{B_n},$$

and

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots = \lim_{n \to +\infty} \frac{A_n}{B_n},$$

where $\{A_n\}$ and $\{B_n\}$ are defined by

$$\begin{cases} A_{-1} = 1, \quad A_0 = b_0, \quad B_{-1} = 0, \quad B_0 = 1, \\ A_n = b_n A_{n-1} + a_n A_{n-2} \quad (n \geq 1) \\ B_n = b_n B_{n-1} + a_n B_{n-2} \quad (n \geq 1) \end{cases} \tag{0.2}$$

Theorem 1. *Let an infinite continued fraction*

$$\alpha = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots \tag{0.3}$$

be convergent, where a_n, b_n $(n \geq 1)$ are non zero rational integers. Assume that

$$\sum_{n=1}^{+\infty} \left| \frac{a_{n+1}}{b_n b_{n+1}} \right| < \infty, \tag{0.4}$$

and

$$\lim_{n \to +\infty} \left| \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n} \right| = 0. \tag{0.5}$$

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Then α is irrational and its irrationality exponent $\mu(\alpha)$ satisfies

$$2+\sigma \le \mu(\alpha) \le 2+\max(\tau_1,\tau_2), \tag{0.6}$$

where

$$\sigma = \limsup_{n\to+\infty} \frac{\log|b_{n+1}| - \log|a_1a_2\cdots a_{n+1}|}{\log|b_1b_2\cdots b_n|}, \tag{0.7}$$

$$\tau_1 = \limsup_{n\to+\infty} \frac{\log|a_1a_2\cdots a_{n+1}|}{\log|b_1b_2\cdots b_n| - \log|a_1a_2\cdots a_n|}, \tag{0.8}$$

and

$$\tau_2 = \limsup_{n\to+\infty} \frac{\log|b_{n+1}| - \log|a_1a_2\cdots a_{n+1}| + 2\log(A_n,B_n)}{\log|b_1b_2\cdots b_n| - \log|a_1a_2\cdots a_n|}$$

with (A_n, B_n) the greatest common divisor of A_n and B_n.

We apply Theorem 1 to continued fractions representing numbers related to Cahen's constant and deduce their transcendence from the obtained lower bounds of their irrationality exponents.

In 1880 Sylvester [11] proved that any real number $0<x<1$ can be expanded uniquely in the series

$$x = \sum_{n=0}^{+\infty} \frac{1}{t_n},$$

where the t_n are integers satisfying the condition $t_0 \ge 2$, $t_{n+1} \ge t_n^2 - t_n + 1$ $(n \ge 0)$, and furthermore that x is irrational if and only if the equality holds for all large n. He examined some of the properties of the (Sylvester) sequence $\{S_n\}_{n\ge0}$ defined by

$$S_0 = 2, \quad S_{n+1} = S_n^2 - S_n + 1 \quad (n \ge 0), \tag{0.9}$$

which satisfies

$$\sum_{n=0}^{+\infty} \frac{1}{S_n} = 1.$$

Cahen [2] and Sierpinski [9] independently obtained similar results for alternating series; namely, any irrational number $0<x<1$ can be uniquely written in the form

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{u_n},$$

where the u_n are integers satisfying $u_0 \ge 1$, $u_{n+1} \ge u_n^2 + u_n$ $(n \ge 0)$. As an example, Cahen [2] mentioned that (Cahen's constant)

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{u_n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{6} - \frac{1}{42} + \frac{1}{1806} - \frac{1}{3263442} + \cdots$$

is an irrational number, where $u_0 = 1$, $u_{n+1} = u_n^2 + u_n$ $(n \ge 0)$, and hence $u_n = S_n - 1$ $(n \ge 0)$. We note that the sequence $\{s_n\}_{n\ge0}$ defined by

$$s_0 = 2, \quad s_{n+1} = s_n^2 + s_n - 1 \quad (n \ge 0) \tag{0.10}$$

satisfies

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{s_n} = \frac{1}{3}.$$

In 1991 Davison and Shallit [4] proved the transcendence of Cahen's constant. Becker [1] generalized the result by Mahler's method.

In this paper we generalize the sequences S_n and s_n defined in (0.9) and (0.10) by introducing the sequences $u_n = u_n(\varepsilon)$ satisfying $u_0 \in \mathbb{N}$, $u_0 > \max(1, \varepsilon)$, and the recurrence

$$u_{n+1} = u_n^2 - \varepsilon u_n + \varepsilon \quad (n \geq 0), \tag{0.11}$$

where ε is a non-zero integer given arbitrary. Next, we define the numbers $\gamma_{l,\varepsilon} = \gamma_{l,\varepsilon}(u_0)$ by

$$\gamma_{l,\varepsilon} = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{\varepsilon^n}{u_n - \varepsilon}\right)^l \quad (l = 1, 2, 3, \cdots). \tag{0.12}$$

We expand the numbers $\gamma_{l,\varepsilon}$ in continued fractions whose partial numerators a_n and denominators b_n satisfy the assumptions in Theorem 1. Applying Theorem 1, we obtain the following

Theorem 2. *Let $\gamma_{l,\varepsilon}$ be the numbers defined by (0.12). Assume that u_0 and ε are coprime. Then $\mu(\gamma_{1,\varepsilon}) = 3$ and*

$$2 + \frac{2}{3l-1} \leq \mu(\gamma_{l,\varepsilon}) \leq 2 + \frac{6(l-1)}{3l+1} \quad (l = 2, 3, 4, \cdots). \tag{0.13}$$

Corollary 1. *For every positive integer l, $\gamma_{l,\varepsilon}$ is a non-Liouville transcendental number.*

We give some examples of the numbers $\gamma_{l,\varepsilon}$.

Example 1. When $\varepsilon = 1$ and $u_0 = 2$, we have

$$\gamma_{l,1}(2) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(S_n - 1)^l} \quad (l = 1, 2, 3, \cdots).$$

In particular, $\gamma_{1,1}(2)$ is Cahen's constant.

Example 2. When $\varepsilon = -1$ and $u_0 = 2$, we obtain

$$\gamma_{l,-1}(2) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(s_n + 1)^l} \quad (l = 2, 4, 6, \cdots),$$

$$\gamma_{l,-1}(2) = \sum_{n=0}^{+\infty} \frac{1}{(s_n + 1)^l} \quad (l = 1, 3, 5, \cdots).$$

Example 3. When $\varepsilon = 2$ and $u_0 = 3$, u_n is the n-th Fermat number:

$$u_n = F_n = 2^{2^n} + 1.$$

Therefore we have

$$\gamma_{l,2}(3) = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{2^n}{F_n - 2}\right)^l = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{2^n}{2^{2^n} - 1}\right)^l \quad (l = 1, 2, 3, \cdots).$$

It should be noted that the irrationality exponent of the sum of the reciprocals of Fermat numbers is equal to 2 (see [3]).

Example 4. Denote by L_n the sequence of Lucas numbers. Define

$$v_n = L_{2^{n+1}} = \Phi^{2^{n+1}} + \Phi^{-2^{n+1}},$$

where $\Phi = \frac{1}{2}\left(1+\sqrt{5}\right)$ is the Golden number. Then clearly $v_{n+1} = v_n^2 - 2$. If we put $u_n = v_n + 2$, we see that $u_0 = 5$ and

$$u_{n+1} = u_n^2 - 4u_n + 4$$

for every $n \geq 0$. Therefore

$$\gamma_{l,4}(5) = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{4^n}{L_{2^{n+1}} - 2}\right)^l \quad (l = 1, 2, 3, \cdots).$$

As for transcendence, much more general results are obtained by Duverney, Kurosawa, and Shiokawa [6]. They discussed transcendence of the values of the series

$$\sum_{n=0}^{\infty} \frac{a^n}{q(p^n(z))}$$

at algebraic points, where $a \in \overline{\mathbb{Q}}, p(z), q(z) \in \overline{\mathbb{Q}}[z]$ with $\deg p(z) \geq 2$ and $\deg q(z) \geq 1$. For example, [6, Example 1.5] states that, if $a \neq 0$ and γ with $S_n \neq \gamma$ for all $n \geq 0$ are algebraic numbers, then

$$\sum_{n=0}^{\infty} \frac{a^n}{(S_n - \gamma)^l},$$

where l is any positive integers, is algebraic if and only if $a = l = 1$ and $\gamma = 0$.

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